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Brief paper

PDE backstepping control of one-dimensional heat equation with time-varying domain ☆

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ABSTRACT

In this work a PDE backstepping-based control law for one-dimensional unstable heat equation with time-varying spatial domain is developed. The underlying parabolic partial differential equation (PDE) with time-varying domain is a model emerging from process control applications such as crystal growth. The use of backstepping control methodology yields the inherent feature of a time-varying PDE describing the kernel of the associated Volterra integral. The well-posedness of PDE kernel is proven and a numerical method to compute the solution of PDE kernel augmented with the error analysis to establish the accuracy of the proposed numerical method is demonstrated. Finally, the explicit form of the full state-feedback control law is given and appropriate simulation is provided for the application of temperature regulation in the Czochralski crystal growth process.

1. Introduction

The model of a wide variety of transport processes can be described in terms of partial differential equations (PDEs) based on the use of theoretical first-principles, experimental studies or system identification. In some applications, the shape of the PDE domain changes due to the phenomena such as phase change, generation and consumption of chemical species through the chemical reaction mechanism, heat and mass transfer. In addition to the time-dependent PDE parameters' nature, the occurrence of these changes introduces more complexities in the modelling, analysis and control of the process dynamics which usually leads to a general class of transport-reaction parabolic PDEs. The representative case of these systems is given by the Czochralski (CZ) crystal growth problem described by the one-dimensional heat parabolic PDE characterized by the presence of the grown crystal boundary velocity as a time-dependent coefficient associated with the first order advective transport term (Derby, Atherton, Thomas, & Brown, 1987).

The methods for control of linear parabolic PDEs with fixed spatial domain by boundary and/or distributed actuation setting are well established (Curtain & Zwart, 1995; Katō, 1995; Krstic & Smyshlyaev, 2008; Lasiecka, 1980). In this realm, there are several contributions which consider a time-varying parabolic PDE with fixed spatial domain (Acquistapace & Terreni, 1987; Pazy, 1983) where solutions to nonautonomous parabolic systems are formulated by two-parameter semigroups which resemble and inherit the properties of the standard one-parameter semigroup generated by time-invariant parabolic operators.

In a large number of cases an analytic expression for the two-parameter semigroup describing the nonautonomous system behaviour cannot be found which prevents direct analysis and controller synthesis. In addition, only few results have been found when it comes to the study of parabolic PDEs with time-varying domain and these results are mainly centred on establishing existence and regularity properties of a solution including the utilization of transformation, which maps the PDE onto a new time invariant spatial domain (Baconneau & Lunardi, 2004; Burdzy, Chen, & Sylvester, 2004; Lunardi, 2004). Among few contributions along this line, Wang (1990) studied the stabilization and optimal control problem of such systems and later on synthesized the linear optimal controller for thermal gradient regulation of crystal growth processes (Wang, 1995). To obtain a reduced-order model of nonlinear parabolic PDE systems with time-varying

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spatial domain, Armaou and Christofides (2001a) used a mathematical transformation to represent the PDE on a time-invariant spatial domain and applied Karhunen–Loève decomposition to find the set of eigenfunctions on the fixed domain. In application, they used this approach in nonlinear feedback (Armaou & Christofides, 2001b) and robust (Armaou & Christofides, 2001c) control of one-dimensional reaction–diffusion systems based on the use of Galerkin's method.

In the last decade, the concept of backstepping emerging from nonlinear finite-dimensional control systems synthesis has been broadened to distributed parameter systems and provided a systematic approach for the boundary controlled linear parabolic PDEs. In the PDE backstepping methodology, a Volterra integral transformation is used to transform the PDE into a suitably selected stable target system. However, the kernel of this transformation is defined by the solution of the kernel PDE that is of a higher-order in space. Having the solution of the kernel PDE, the state-feedback control law which embeds desired transformation is obtained (Krstic & Smyshlyaev, 2008). The form of the hyperbolic kernel PDE is recognized in physics as Klein–Gordon equation. The powerful technique of backstepping synthesis provides a framework to handle a large class of distributed parameter systems controlled at the boundary. Namely, the stabilization of parabolic PDEs with spatially varying coefficients and advection–diffusion type of parabolic PDEs is easily realized by this method (Smyshlyaev & Krstic, 2004), as well as the boundary stabilization of the wave equation (Krstic, Guo, Balogh, & Smyshlyaev, 2008). The introduction of the backstepping approach goes back to the works of Liu (2003) and Smyshlyaev and Krstic (2004) in the design of stabilizing state-feedback controllers, the reader is referred to Ref. (Krstic & Smyshlyaev, 2008) for the list.

The main objective of this work is to synthesize a controller for PDE systems with time-varying spatial domain using backstepping approach, whereas the previous works considered fixed domains. The time-varying nature of PDE systems with moving boundaries introduces time-dependent parameters in the distributed system description. Along this way, Smyshlyaev and Krstic (2005) studied the backstepping control of one-dimensional unstable heat equation with space-dependent diffusion and time-dependent reaction parameters. The form of the integral transformation obtained in the synthesis of an explicit controller that aims to stabilize parabolic PDEs with time-varying reaction term leads to the time-varying kernel PDE which is defined over a fixed domain. In addition to this case, another instance of time-varying kernels incorporated in Volterra-type integral transformations is used in the stability analysis of time-varying input and state delays for nonlinear systems (Bekiaris-Liberis & Krstic, 2012; Krstic, 2010). Finally, the works of Meurer and Kugi (2009) on the design of a tracking controller and Meurer (2013) on the extension of Luenberger-type observers for semilinear PDEs broadened the PDE backstepping approach to the systems with time-dependent parameters.

The moving boundary aspect of the problem we are interested in suggests that not only time-varying parameters are associated with the characterization of the PDE system, but also the transformation kernel function is described on the time-varying domain. In the following, the formulation to the PDE backstepping boundary control of an unstable one-dimensional heat equation described on a domain with moving boundaries is presented. Initially, we establish stability of the target system by invoking a form of Poincaré inequality. Then, the PDE system is transformed to an exponentially stable target system through a Volterra-type integral transformation to obtain the PDE describing the transformation kernel. The kernel PDE on the nontrivial time-varying domain is then transformed to a fixed domain and subsequently, analysis and a numerical solution to the kernel PDE is presented. An explicit full state feedback control law and appropriate simulations are given to demonstrate successful stabilization of the unstable system.

2. The control problem

Consider the linear one-dimensional parabolic PDE system of the form:

$$\partial_t u(x,t) = \alpha \partial_x^2 u(x,t) - \dot{l}(t)\partial_x u(x,t) + \lambda_0 u(x,t) \tag{1}$$

where $u(x,t)$ is the state variable, $\mathbb{D}(t) = [0, l(t)] \subset \mathbb{R}$ is the time-varying domain of the definition of PDE, $x \in \mathbb{D}(t)$ is the spatial coordinate, $l(t) \in \mathbb{R}^+$ is the smooth time-dependent function describing the length of the domain, $\dot{l}(t)$ is its bounded time derivative, and $t \in [0, \infty)$ is the time. $\alpha > 0$ and λ_0 are process parameters and the advection term appearing in (1) is due to the moving boundaries of the PDE domain (Derby et al., 1987; Izadi & Dubljevic, 2013). The jth partial derivative of a multivariable function ϕ with respect to the variable ζ is denoted by $\partial_\zeta^j \phi$, for the first derivative the superscript is dropped for simplicity. The temperature distribution in the crystal of the Czochralski process or the shrinkage of a catalyst in a chemical process are two typical examples of the models that can be approximated by this PDE system. Although this approach can be generalized for standard types of boundary conditions, we assume the following boundary setting for the PDE system which is applicable to the temperature stabilization in the Czochralski crystal growth process:

$$\begin{aligned} &u(0,t) = 0 \\ &\partial_x u(l(t),t) = U(t). \end{aligned} \tag{2}$$

Here $U(t)$ is the control applied at the boundary $x = l(t)$ to stabilize the system state.

Remark 1. The invertible transformations of space $\zeta = x/l(t)$ and time $\tau = \int_0^t l^{-2}(s)ds$ represent the system (1)–(2) as:

$$\begin{aligned} \partial_\tau u(\zeta,\tau) \;&=\; \alpha\partial_\zeta^2 u(\zeta,\tau) + \frac{dl(t(\tau))}{dt} l(t(\tau))(\zeta - 1)\partial_\zeta u(\zeta,\tau) \\ &\quad + l^2(t(\tau))\lambda_0 u(\zeta,\tau) \\ u(0,\tau) &= 0 \\ \frac{\partial_\zeta u(1,\tau)}{l(t(\tau))} &= U(t(\tau)) \end{aligned}$$

on the fixed domain $\zeta = [0, 1]$. This approach is called boundary immobilization method (BIM) and the available formulation (Meurer & Kugi, 2009) can be used for the transformed system. In application, the length of the time-varying domain $l(t)$ is not known *a priori* and needs to be measured at each time instance. Noisy measurement of this parameter questions the robustness of transforming PDE to a fixed domain and this method is not followed in this work.

In order to study the stability of the target system and specify boundedness and differentiability of the kernel function, the following assumption is required.

Assumption 2. The function $l(t)$ is analytic, or alternatively, there exists a real positive constant $\bar{D}$ such that for every non-negative integer j the following bound holds:

$$|\partial_t^j l(t)| \le \bar{D}^{j+1} j! \tag{3}$$

In the subsequent sections, the plant (1)–(2) will be transformed to the target system:

$$\partial_t w(x,t) = \alpha\partial_x^2 w(x,t) - c w(x,t) \tag{4}$$

$$\begin{cases} w(0,t) = 0 \\ \partial_x w(l(t),t) = -\dfrac{\dot{l}(t)}{2\alpha} w(l(t),t) \end{cases} \tag{5}$$

with constant $c \geq 0$. The well-posedness of PDEs of this type with any initial condition $w(x,0) = w_0(x) \in L_2(\mathbb{D}(0))$ is shown by Ng and Dubljevic (2012). To show the stability of this system, first we need the following lemma which can be proved easily with the use of Cauchy–Schwarz and Young's inequalities.

Lemma 3. *The following conservative form of Poincaré inequality holds for the time-varying space $\mathbb{D}(t)$:*

$$\int_0^{l(t)} w^2(x,t)dx \leq 2l(t)w^2(0,t) + 4l^2(t)\int_0^{l(t)} (\partial_x w(x,t))^2\, dx. \quad (6)$$

Now define the L_2-norm on the time-varying space $\mathbb{D}(t)$ as:

$$\|w(x,t)\| = \left(\int_0^{l(t)} w^2(x,t)dx\right)^{\frac{1}{2}}. \quad (7)$$

The stability of the target system is explored in the following lemma.

Lemma 4. *The PDE system (4)–(5) is exponentially stable in the sense of the L_2-norm.*

Proof. Consider the Lyapunov function $V(t) = \frac{1}{2}\|w(x,t)\|^2$, its time derivative is:

$$\begin{aligned}\dot{V}(t) &= \int_0^{l(t)} w(x,t)\partial_t w(x,t)dx + \frac{1}{2}\dot{l}(t)w^2(l(t),t) \\ &= \int_0^{l(t)} (\alpha w(x,t)\partial_x^2 w(x,t) - cw^2(x,t))dx + \frac{1}{2}\dot{l}(t)w^2(l(t),t) \\ &= -\alpha w(0,t)\partial_x w(0,t) + \alpha w(l(t),t)\partial_x w(l(t),t) \\ &\quad + \int_0^{l(t)} [-\alpha(\partial_x w(x,t))^2 - cw^2(x,t)]dx + \frac{1}{2}\dot{l}(t)w^2(l(t),t).\end{aligned}$$

The first term vanishes from the first boundary condition in (5) and the second and last terms cancel out from the second boundary condition, leading to the following:

$$\dot{V}(t) = -\alpha\int_0^{l(t)} (\partial_x w(x,t))^2 dx - c\int_0^{l(t)} w^2(x,t)dx.$$

Imposing bound $l(t) \leq \bar{D}$ from Assumption 2 to the Poincaré inequality (6), we have:

$$\dot{V}(t) \leq -\left(\tfrac{\alpha}{4\bar{D}^2} + c\right)\int_0^{l(t)} w^2(x,t)dx = -\left(\tfrac{\alpha}{2\bar{D}^2} + 2c\right)V(t)$$

which implies the exponential stability of the target system in the L_2-norm. □

3. Mathematical transformations

To eliminate advection term in (1), the transformation

$$u(x,t) = v(x,t)e^{\int_0^x \frac{\dot{l}(t)}{2\alpha}dy} = v(x,t)e^{\frac{\dot{l}(t)x}{2\alpha}} \quad (8)$$

is used, which yields to the following plant

$$\partial_t v(x,t) = \alpha\partial_x^2 v(x,t) + \lambda(x,t)v(x,t) \quad (9)$$

$$\begin{cases} v(0,t) = 0 \\ \left[\partial_x v(l(t),t) + \dfrac{\dot{l}(t)}{2\alpha}v(l(t),t)\right]e^{\frac{\dot{l}(t)l(t)}{2\alpha}} = U(t) \end{cases} \quad (10)$$

where $\lambda(x,t) = \lambda_0 - \frac{\dot{l}^2(t)+2\ddot{l}(t)x}{4\alpha}$. Note that the stability of $v(x,t)$ in the L_2-norm implies the stability of $u(x,t)$ since the norm $\|e^{\frac{\dot{l}(t)x}{2\alpha}}\|$ is bounded for all t.

The moving boundary characteristic of the PDE system imposes the use of the Volterra integral transformation given as follows:

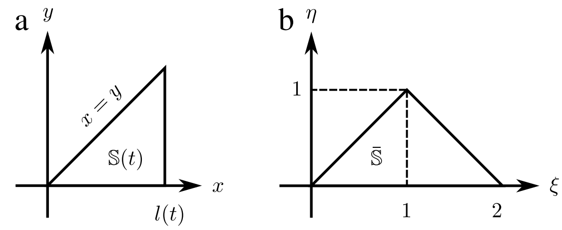

Fig. 1. (a) Time-dependent domain $\mathbb{S}(t)$ of the kernel function $k(x,y,t)$ and (b) fixed domain $\bar{\mathbb{S}}$ of the kernel function $k(\xi,\eta,t)$.

$$w(x,t) = v(x,t) - \int_0^x k(x,y,t)v(y,t)dy \quad (11)$$

which maps (9)–(10) into the stable target system (4)–(5) with a free parameter c to manipulate the system's response. Substitution of (11) into (4)–(5) and use of (9)–(10) yields to the following time-varying PDE for the kernel function $k(x,y,t)$:

$$\begin{aligned}\partial_t k(x,y,t) &= \alpha(\partial_x^2 k(x,y,t) - \partial_y^2 k(x,y,t)) \\ &\quad - (\lambda(y,t) + c)k(x,y,t) \quad (12)\end{aligned}$$

$$\begin{cases} k(x,0,t) = 0 \\ k(x,x,t) = -\dfrac{1}{2\alpha}\displaystyle\int_0^x (\lambda(y,t) + c)dy \end{cases} \quad (13)$$

defined on the time-varying domain $\mathbb{S}(t) = \{(x,y)|0 \leq y \leq x \leq l(t)\} \subset \mathbb{R}^2$ shown in Fig. 1. Hence, the control problem (1)–(2) is now reduced to finding the solution of (12)–(13). Also, the control action can be found in the form of a state-feedback as:

$$\begin{aligned}U(t) = \Bigg(&\int_0^{l(t)} \left[\frac{\dot{l}(t)}{2\alpha}k(l(t),y,t) + \partial_x k(l(t),y,t)\right]v(y,t)dy \\ &+ k(l(t),l(t),t)v(l(t),t)\Bigg)e^{\frac{\dot{l}(t)l(t)}{2\alpha}}. \quad (14)\end{aligned}$$

Finding the solution to (12)–(13) on the time-dependent domain $\mathbb{S}(t)$ is not straight forward. However, the following transformation will map $\mathbb{S}(t)$ to the fixed domain $\bar{\mathbb{S}} = \{(\xi,\eta)|0 \leq \eta \leq \xi \leq 2-\eta\} \subset \mathbb{R}^2$ (see Fig. 1(b)):

$$\xi = \frac{x+y}{l(t)}, \qquad \eta = \frac{x-y}{l(t)}. \quad (15)$$

Using (15), the kernel PDE for $k(\xi,\eta,t)$ takes the form:

$$\begin{aligned}\partial_t k(\xi,\eta,t) &= \frac{4\alpha}{l^2(t)}\partial_\xi\partial_\eta k(\xi,\eta,t) + \frac{\dot{l}(t)}{l(t)}(\xi\partial_\xi k(\xi,\eta,t) \\ &\quad + \eta\partial_\eta k(\xi,\eta,t)) - \bar{\lambda}(\xi,\eta,t)k(\xi,\eta,t) \quad (16)\end{aligned}$$

$$\begin{cases} k(\xi,\xi,t) = 0 \\ k(\xi,0,t) = f(\xi,t) \end{cases} \quad (17)$$

where

$$\begin{aligned}\bar{\lambda}(\xi,\eta,t) &= \lambda\left(\frac{l(t)}{2}(\xi-\eta),t\right) + c \\ &= -\frac{\dot{l}^2(t) + l(t)\ddot{l}(t)(\xi-\eta)}{4\alpha} + \lambda_0 + c\end{aligned}$$

$$f(\xi,t) = \frac{\xi l(t)}{4\alpha}\left[\frac{2\dot{l}^2(t) + \xi l(t)\ddot{l}(t)}{8\alpha} - (\lambda_0 + c)\right].$$

4. Analysis of the kernel PDE

In general, the solution to the kernel PDE (16)–(17) is found by means of successive integration. Although this method is computationally rather expensive except for special cases, it is a useful tool to analyse the well-posedness of the kernel PDE. To follow this method, integrate (16) with respect to η from 0 to η and then with respect to ξ from η to ξ and use boundary conditions (17) and integration by-parts for the terms with the first-order derivative to transform the kernel PDE (16)–(17) into the following integral equation:

$$\int_\eta^\xi \int_0^\eta \left[\partial_t k(\rho,\sigma,t) + \left(2\frac{\dot{l}(t)}{l(t)} + \bar{\lambda}(\rho,\sigma,t)\right) k(\rho,\sigma,t)\right] d\sigma d\rho$$
$$-\frac{\dot{l}(t)}{l(t)}\left(\xi \int_0^\eta k(\xi,\sigma,t)d\sigma + \eta\int_\eta^\xi k(\rho,\eta,t)d\rho\right)$$
$$-\frac{4\alpha}{l^2(t)}\left[k(\xi,\eta,t) - f(\xi,t) + f(\eta,t)\right] = 0 \tag{18}$$

that is rewritten as:

$$k(\xi,\eta,t) = K_1(\xi,\eta,t) + \mathcal{G}k(\xi,\eta,t) \tag{19}$$

where

$$K_1(\xi,\eta,t) = f(\xi,t) - f(\eta,t) \tag{20}$$
$$\mathcal{G}k(\xi,\eta,t) = \frac{l^2(t)}{4\alpha}\int_\eta^\xi\int_0^\eta \partial_t k(\rho,\sigma,t)d\sigma d\rho$$
$$+\int_\eta^\xi\int_0^\eta\left(\frac{\dot{l}(t)l(t)}{2\alpha} + \frac{l^2(t)\bar{\lambda}(\rho,\sigma,t)}{4\alpha}\right)k(\rho,\sigma,t)d\sigma d\rho$$
$$-\frac{\dot{l}(t)l(t)}{4\alpha}\left(\xi\int_0^\eta k(\xi,\sigma,t)d\sigma + \eta\int_\eta^\xi k(\rho,\eta,t)d\rho\right).$$

The method of successive integration suggests that the solution to the integral equation (18) is given by the series

$$k(\xi,\eta,t) = \sum_{n=1}^{\infty} K_n(\xi,\eta,t) \tag{21}$$
$$K_{n+1}(\xi,\eta,t) = \mathcal{G}K_n(\xi,\eta,t) \tag{22}$$

with $K_1(\xi,\eta,t)$ given in (20). The convergence of this series and hence, the existence of a solution for (18) is given by the following theorem:

Theorem 5. *The jth time-derivative of $K_n(\xi,\eta,t)$ is bounded by*

$$\sup_t \left|\partial_t^j K_n(\xi,\eta,t)\right| \le \frac{3^{n+1}\gamma_{j,n}D^{j+2n}(j+n-1)!}{(n-1)!(n-1)!n!}(\xi\eta)^{n-1} \tag{23}$$

where $D \ge 1$ is a real constant and

$$\gamma_{j,n} = \frac{\Gamma\left(\frac{4n+j}{3}+1\right)}{\Gamma\left(\frac{n+j}{3}+1\right)} \tag{24}$$

with $\Gamma(z)$ being the gamma function. Moreover, the series (21) is absolutely convergent.

The proof to theorem follows the same methodology as in Meurer and Kugi (2009) and it is provided in Appendix A. Note that $K_n(\xi,\eta,t)$ is $C^2(\bar{\mathbb{S}}\times[0,\infty))$ which follows from (20) and (22). Therefore, $k(\xi,\eta,t)$ and $k(x,y,t)$ are also $C^2(\bar{\mathbb{S}}\times[0,\infty))$ and $C^2(\mathbb{S}\times[0,\infty))$, respectively, the latter is the result of the invertibility of transformation (15).

Lemma 6. *The control (14) given by the solution of (12)–(13) stabilizes the PDE system (9)–(10) in the L_2-norm.*

Proof. Consider the inverse backstepping transformation in terms of

$$v(x,t) = w(x,t) + \int_0^x q(x,y,t)w(y,t)dy \tag{25}$$

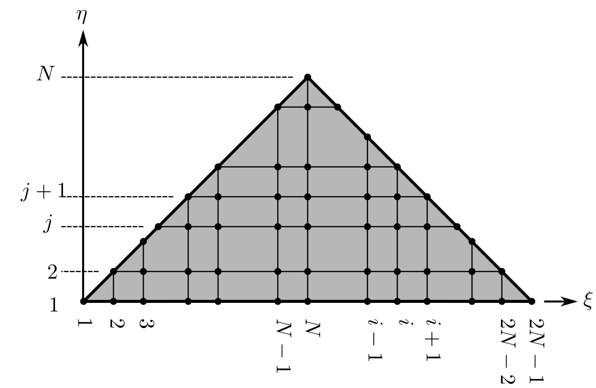

Fig. 2. Discretization of the kernel PDE domain.

with the kernel $q(x,y,t)$. Substitution of (25) into (9)–(10) and use of (4)–(5) results in the following PDE for the inverse kernel function:

$$\partial_t q(x,y,t) = \alpha(\partial_x^2 q(x,y,t) - \partial_y^2 q(x,y,t)) + (\lambda(x,t)+c)q(x,y,t) \tag{26}$$

$$\begin{cases} q(x,0,t) = 0 \\ q(x,x,t) = -\dfrac{1}{2\alpha}\displaystyle\int_0^x (\lambda(y,t)+c)dy. \end{cases} \tag{27}$$

This PDE is similar to (12)–(13) and Theorem 5 shows the existence of a solution. Hence, the transformation (11) is invertible and the stability of the target system (4)–(5) implies the stability of the closed-loop system (9)–(10). □

Theorem 7. *For any initial condition $v_0(x) \in L_2(\mathbb{D}(0))$ that satisfies (10) with $U(t)$ given in (14) for $t = 0$, the PDE system (9)–(10) with boundary control (14) has a unique solution.*

The proof this theorem is given by Ng and Dubljevic (2012) where the closed-loop system (9)–(10) is represented as a nonautonomous parabolic evolution system with solutions given by a two-parameter evolution operator, following the boundedness and continuity of the kernel function $k(x,y,t)$.

5. Numerical method

To find an approximate solution to the integral equation (18), we use the numerical method introduced in Jadachowski, Meurer, and Kugi (2012). This approach is computationally tractable and is based on the approximation of integrals by the use of a composite trapezoidal rule. To this end, the triangular spatial domain is discretized in N^2 computational points on an equally-spaced square grid as shown in Fig. 2. Hence, each continuous function $g(\xi,\eta,t)$ is discretized and denoted by $g_{i,j}(t)$ at the coordinate (ξ_i,η_j) or simply at the point $(i,j) \in \{(m,n)|1 \le n \le m \le 2N-n\} \subset \mathbb{N}^2$. Now, the integrals in (18) are approximated by using discrete values of their integrands evaluated at grid points through the application of the composite trapezoidal rule as:

$$\int_{\eta_J}^{\xi_I}\int_0^{\eta_I} g_{I,J}(t)d\eta d\xi$$
$$\approx \frac{\Delta^2}{4}\sum_{i=J}^{I-1}\sum_{j=1}^{J-1}\left[g_{i,j}(t) + g_{i+1,j}(t) + g_{i,j+1}(t) + g_{i+1,j+1}(t)\right] \tag{28}$$

where $\Delta = 1/(N-1)$. Using proper indexing to represent the discretized kernel function $\bar{k}_{i,j}(t)$ as the vector $\kappa_r(t)$, the integral equation will reduce in the following point-wise equation (see Appendix B):

$$A\dot{\kappa}(t) + B(t)\kappa(t) + H(t) = 0. \tag{29}$$

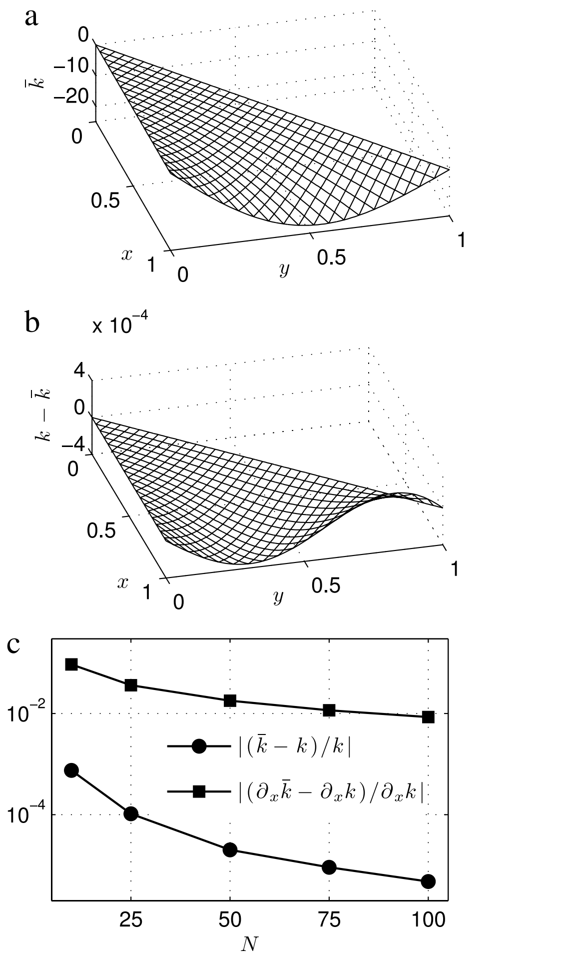

Fig. 3. (a) Approximated solution to the kernel PDE for $l(t) = 1$ and (b) associated approximation error. (c) Numerical error of the kernel function and its derivative at $(x, y) = (1, 0.5)$ on the actuation boundary for different discretization levels defined by N.

The initial condition $\kappa_0 = \kappa(0)$ for (29) is chosen as the stationary solution at $t = 0$:

$$\kappa(0) = -B(0)^{-1}H(0). \tag{30}$$

Note that (29)–(30) form an initial-value problem as a set of linear time-varying ordinary differential equations and can be solved efficiently using available numerical methods. The over bar in $\bar{k}_{i,j}(t)$ indicates the approximated value of the kernel function.

6. Simulation results

For the special case $\alpha = 1, c = 0$ and fixed domain $l(t) = 1$, there is a closed-form solution for the kernel function (Krstic & Smyshlyaev, 2008) as:

$$k(x, y) = -\lambda_0 y \frac{I_1(\sqrt{\lambda_0(x^2 - y^2)})}{\sqrt{\lambda_0(x^2 - y^2)}} \tag{31}$$

where $I_1(x)$ is the first-order modified Bessel function of the first kind. This solution can be used to validate and assess the accuracy of the numerical method used to solve the kernel PDE. Fig. 3 shows the approximated kernel function and associated numerical error of the solution as well as the numerical approximation of the kernel function $k(1, 0.5)$ and its derivative $\partial_x k(1, 0.5)$ for $\lambda_0 = 20$. The differentiation of the kernel function is performed by finite differencing and since the derivative term appears in the expression for control (14), it is included in the error analysis.

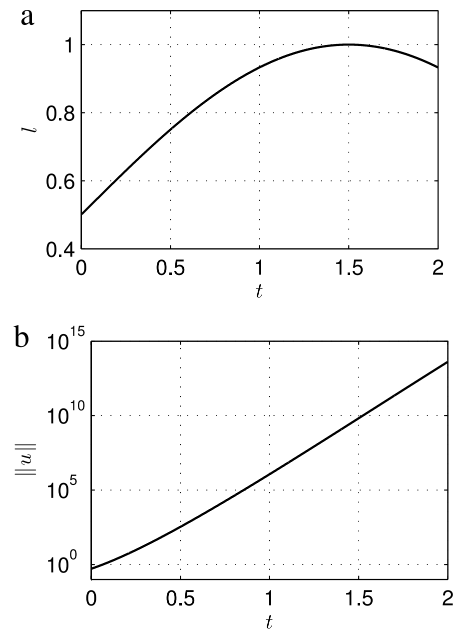

Fig. 4. (a) Length $l(t)$ of the time-varying domain of the PDE system. (b) The norm of the state of uncontrolled system for the given parameters.

We choose the discretization associated with $N = 75$ for the error of order 0.01 for the kernel function derivative in the following simulations. The numerical approximation of the time-varying kernel function is used to stabilize the system given by (1)–(2) with $\alpha = 1$ and $\lambda_0 = 20$ on the time-varying domain $\mathbb{D}(t)$ shown in Fig. 4(a). For these process parameters, the PDE system is unstable as depicted by the evolution of the norm of the system as given in Fig. 4(b).

The evolution of approximate solution to the kernel PDE on the (ξ, η)-domain for $c = 0$ is shown in Fig. 5. Fig. 6(a) shows a closed-loop response of the system as the evolution of the state for an arbitrary chosen initial condition. Finally, control input and evolution of the L_2-norm of the state are shown in Fig. 6(b). The stabilization of the unstable PDE system is well provided in the simulations.

7. Summary

The PDE backstepping boundary control synthesis of one-dimensional heat equation on a time-varying domain is formulated in this work. The PDE system is transformed to an exponentially stable target system through the invertible Volterra-type integral transformation resulting in the two-dimensional time-varying PDE with time-dependent domain describing the transformation kernel. Then, a numerical solution to the kernel PDE is provided and simulated to demonstrate stabilization of the unstable system with time-varying domain.

Appendix A. Proof to Theorem 5

With Assumption 2, all the functions $\frac{l(t)\dot{l}^2(t)}{16\alpha^2}$, $\frac{l^2(t)\ddot{l}(t)}{32\alpha^2}$, $\frac{l(t)(\lambda_0+c)}{4\alpha}$, $\frac{l^2(t)}{4\alpha}$, $\frac{l(t)\dot{l}(t)}{2\alpha} + \frac{l^2(t)\dot{\lambda}}{4\alpha}$ are analytic as well. Hence, there exists real constant $D \geq 1$ such that the jth time-derivative of these functions is bounded by $D^{j+1}j!$ for $j = 0, 1, 2, \ldots$.

Because of the appearance of the time-derivative of $K_n(\xi, \eta, t)$, the convergence analysis of series (21) requires to find a bound for $\left|\partial_t^j K_n(\xi, \eta, t)\right|$, which is given in (23) and is proved here by induction. For $n = 1$:

$$\sup_t \left|\partial_t^j K_1(\xi, \eta, t)\right| = \sup_t \left|\partial_t^j \left[f(\xi, t) - f(\eta, t)\right]\right|$$

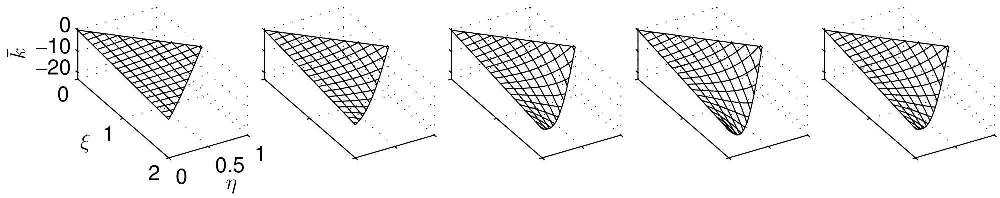

Fig. 5. Approximated kernel function $\bar{k}(\xi, \eta, t)$ at $t = 0, 0.5, 1, 1.5$ and 2.

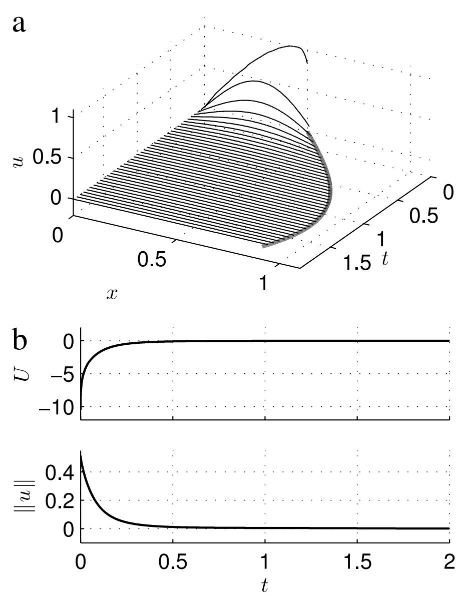

Fig. 6. (a) Closed-loop profile of $u(x, t)$. The thick grey line shows evolution of domain. (b) Control input and evolution of the norm of the state.

$$\begin{aligned}
&= \sup_t \left| \partial_t^j \left[\frac{l(t)\dot{l}^2(t)}{16\alpha^2}(\xi - \eta) + \frac{l^2(t)\ddot{l}(t)}{32\alpha^2}(\xi^2 - \eta^2) \right.\right. \\
&\quad \left.\left. - \frac{l(t)(\lambda_0 + c)}{4\alpha}(\xi - \eta) \right] \right| \\
&\le D^{j+1} j! \left(|\xi - \eta| + \left|\xi^2 - \eta^2\right| + |\xi - \eta| \right) \\
&\le 8D^{j+1} j! \le 9\gamma_{j,1} D^{j+2} j!
\end{aligned}$$

which is equal to the right-hand side of (23) for $n = 1$. Assuming (23) holds for all $n = 1, 2, \ldots, N$, the upper bound of $\left|\partial_t^j K_{N+1}(\xi, \eta, t)\right|$ can be determined as given in Box I which is identical to right-hand side of (23) for $n = N + 1$. This completes the proof of (23). In this sequence of inequalities, we used the following relations for $(\xi, \eta) \in \bar{\mathbb{S}}$, $j = 0, 1, 2, \ldots$ and $n = 1, 2, \ldots$:

$$\begin{aligned}
&|\xi - \eta| \le 2 \quad \text{and} \quad \left|\xi^2 - \eta^2\right| \le 4 \\
&\int_\eta^\xi \int_0^\eta (\rho\sigma)^{n-1} d\sigma d\rho = \frac{\eta^n(\xi^n - \eta^n)}{n^2} \le \frac{(\xi\eta)^n}{n^2} \\
&\xi \int_0^\eta (\xi\sigma)^{n-1} d\sigma = \frac{(\xi\eta)^n}{n} \\
&\eta \int_\eta^\xi (\rho\eta)^{n-1} d\rho = \frac{\eta^n(\xi^n - \eta^n)}{n} \le \frac{(\xi\eta)^n}{n} \\
&\gamma_{j,1} = \frac{j+1}{3} + 1 \ge 1 \\
&\gamma_{i,n} \le \gamma_{j,n} \quad \text{for } i \le j
\end{aligned}$$

$$\begin{aligned}
\gamma_{j+1,n}(4n + j + 4) &= 3\frac{\Gamma\left(\frac{4n+j+1}{3} + 1\right)}{\Gamma\left(\frac{n+j+1}{3} + 1\right)} \frac{(4n + j + 4)}{3} \\
&= 3\frac{\Gamma\left(\frac{4n+j+4}{3} + 1\right)}{\Gamma\left(\frac{n+j+1}{3} + 1\right)} = 3\gamma_{j,n+1}
\end{aligned}$$

$$\sum_{i=0}^{j} \binom{j}{i} (j - i)!(i + n)! = \frac{(j + n + 1)!}{n + 1}.$$

Let $j = 0$ in (23) to find $\sup_t |K_n(\xi, \eta, t)| \le 3\frac{(3D^2)^n \gamma_{0,n}}{(n-1)!n!}(\xi\eta)^{n-1}$ which provides the following bound for series (21):

$$|k(\xi, \eta, t)| \le \sum_{n=1}^{\infty} |K_n(\xi, \eta, t)| \le \sum_{n=1}^{\infty} 3a_n(\xi, \eta)$$

with $a_n(\xi, \eta) = \frac{(3D^2)^n \gamma_{0,n}}{(n-1)!n!}(\xi\eta)^{n-1}$. Hence, the ratio $|a_{n+1}(\xi, \eta)/a_n(\xi, \eta)|$ is

$$\frac{3D^2\xi\eta}{n(n + 1)} \frac{\gamma_{0,n+1}}{\gamma_{0,n}}.$$

Having the limit

$$\begin{aligned}
\lim_{n\to\infty} \frac{\gamma_{0,n+1}}{\gamma_{0,n}} &= \lim_{n\to\infty} \frac{\Gamma\left(\frac{4n}{3} + \frac{7}{3}\right)\Gamma\left(\frac{n}{3} + 1\right)}{\Gamma\left(\frac{4n}{3} + 1\right)\Gamma\left(\frac{n}{3} + \frac{4}{3}\right)} \\
&= \lim_{n\to\infty} \frac{\Gamma\left(\frac{4n}{3}\right)\left(\frac{4n}{3}\right)^{\frac{7}{3}}\Gamma\left(\frac{n}{3}\right)\frac{n}{3}}{\Gamma\left(\frac{4n}{3}\right)\frac{4n}{3}\Gamma\left(\frac{n}{3}\right)\left(\frac{n}{3}\right)^{\frac{4}{3}}} \\
&= \lim_{n\to\infty} \left(\frac{4n}{3}\right)^{\frac{4}{3}} \left(\frac{n}{3}\right)^{\frac{-1}{3}} = \lim_{n\to\infty} \frac{4\sqrt[3]{4}}{3} n
\end{aligned}$$

the series $\sum_{n=1}^{\infty} a_n(\xi, \eta)$ is convergent since $\lim_{n\to\infty} |a_{n+1}(\xi, \eta) / a_n(\xi, \eta)| = \lim_{n\to\infty} \frac{4\sqrt[3]{4}D^2\xi\eta}{n+1} = 0 < 1$, therefore the series (21) is absolutely convergent by comparison.

Appendix B. Discretization of the kernel integral equation

To discretize (18), the integrals are replaced by composite trapezoidal rule as in (28) to obtain:

$$\begin{aligned}
&\frac{\Delta^2}{4} \sum_{i=J}^{I-1} \sum_{j=1}^{J-1} \left(\dot{\bar{k}}_{i,j} + \dot{\bar{k}}_{i+1,j} + \dot{\bar{k}}_{i,j+1} + \dot{\bar{k}}_{i+1,j+1} \right) \\
&+ \frac{\Delta^2}{4} \left(2\frac{\dot{l}(t)}{l(t)} + c \right) \sum_{i=J}^{I-1} \sum_{j=1}^{J-1} \left(\bar{k}_{i,j} + \bar{k}_{i+1,j} + \bar{k}_{i,j+1} + \bar{k}_{i+1,j+1} \right) \\
&+ \frac{\Delta^2}{4} \sum_{i=J}^{I-1} \sum_{j=1}^{J-1} \left(\bar{\lambda}_{i,j}\bar{k}_{i,j} + \bar{\lambda}_{i+1,j}\bar{k}_{i+1,j} \right. \\
&+ \left. \bar{\lambda}_{i,j+1}\bar{k}_{i,j+1} + \bar{\lambda}_{i+1,j+1}\bar{k}_{i+1,j+1} \right) \\
&- \frac{\Delta}{2} \frac{\dot{l}(t)}{l(t)} \left[\xi_I \sum_{j=1}^{J-1} \left(\bar{k}_{I,j} + \bar{k}_{I,j+1} \right) + \eta_J \sum_{i=J}^{I-1} \left(\bar{k}_{i,J} + \bar{k}_{i+1,J} \right) \right] \\
&- \frac{4\alpha}{l^2(t)} \left[\bar{k}_{I,J} - f(\xi_I, t) + f(\eta_J, t) \right] = 0. \qquad \text{(B.1)}
\end{aligned}$$

$$
\begin{aligned}
\sup_t \left|\partial_t^j K_{N+1}(\xi,\eta,t)\right| \le\ & \sup_t \left|\partial_t^j \mathcal{G} K_N(\xi,\eta,t)\right| \le \sup_t \left|\sum_{i=0}^{j}\binom{j}{i}\partial_t^{j-i}\left(\frac{l^2(t)}{4\alpha}\right)\int_\eta^\xi\int_0^\eta \partial_t^{i+1}K_N(\rho,\sigma,t)d\sigma d\rho\right| \\
& + \sup_t \left|\sum_{i=0}^{j}\binom{j}{i}\int_\eta^\xi\int_0^\eta \partial_t^{j-i}\left(\frac{\dot{l}(t)l(t)}{2\alpha}+\frac{l^2\bar{\lambda}(\rho,\sigma,t)}{4\alpha}\right)\partial_t^i K_N(\rho,\sigma,t)d\sigma d\rho\right| \\
& + \sup_t \left|\sum_{i=0}^{j}\binom{j}{i}\partial_t^{j-i}\left(\frac{\dot{l}(t)l(t)}{4\alpha}\right)\xi\int_0^\eta \partial_t^i K_N(\xi,\sigma,t)d\sigma\right| \\
& + \sup_t \left|\sum_{i=0}^{j}\binom{j}{i}\partial_t^{j-i}\left(\frac{\dot{l}(t)l(t)}{4\alpha}\right)\eta\int_\eta^\xi \partial_t^i K_N(\rho,\eta,t)d\rho\right| \\
\le\ & \sum_{i=0}^{j}\binom{j}{i}D^{j-i+1}(j-i)!\frac{3^{N+1}\gamma_{i+1,N}D^{i+2N+1}(i+N)!}{(N-1)!(N-1)!N!}\left|\int_\eta^\xi\int_0^\eta(\rho\sigma)^{N-1}d\sigma d\rho\right| \\
& + \sum_{i=0}^{j}\binom{j}{i}D^{j-i+1}(j-i)!\frac{3^{N+1}\gamma_{i,N}D^{i+2N}(i+N-1)!}{(N-1)!(N-1)!N!}\left|\int_\eta^\xi\int_0^\eta(\rho\sigma)^{N-1}d\sigma d\rho\right| \\
& + \sum_{i=0}^{j}\binom{j}{i}D^{j-i+1}(j-i)!\frac{3^{N+1}\gamma_{i,N}D^{i+2N}(i+N-1)!}{(N-1)!(N-1)!N!}\left|\xi\int_0^\eta(\xi\sigma)^{N-1}d\sigma\right| \\
& + \sum_{i=0}^{j}\binom{j}{i}D^{j-i+1}(j-i)!\frac{3^{N+1}\gamma_{i,N}D^{i+2N}(i+N-1)!}{(N-1)!(N-1)!N!}\left|\eta\int_\eta^\xi(\rho\eta)^{N-1}d\rho\right| \\
\le\ & \frac{3^{N+1}\gamma_{j+1,N}D^{j+2N+2}}{(N-1)!(N-1)!N!}\frac{(\xi\eta)^N}{N^2}\sum_{i=0}^{j}\binom{j}{i}(j-i)!(i+N)! \\
& + \frac{3^{N+1}\gamma_{j,N}D^{j+2N+1}}{(N-1)!(N-1)!N!}\frac{(\xi\eta)^N}{N^2}\sum_{i=0}^{j}\binom{j}{i}(j-i)!(i+N-1)! \\
& + \frac{3^{N+1}\gamma_{j,N}D^{j+2N+1}}{(N-1)!(N-1)!N!}\frac{(\xi\eta)^N}{N}\sum_{i=0}^{j}\binom{j}{i}(j-i)!(i+N-1)! \\
& + \frac{3^{N+1}\gamma_{j,N}D^{j+2N+1}}{(N-1)!(N-1)!N!}\frac{(\xi\eta)^N}{N}\sum_{i=0}^{j}\binom{j}{i}(j-i)!(i+N-1)! \\
\le\ & \frac{3^{N+1}D^{j+2N+2}(j+N)!}{(N-1)!N!N!}(\xi\eta)^N\left[\frac{\gamma_{j+1,N}(j+N+1)}{N(N+1)}+\frac{\gamma_{j,N}}{N^2}+2\frac{\gamma_{j,N}}{N}\right] \\
\le\ & \frac{3^{N+1}D^{j+2N+2}(j+N)!}{N!N!N!}(\xi\eta)^N\left[\frac{\gamma_{j+1,N}(j+N+1)}{N+1}+3\gamma_{j+1,N}\right] \\
=\ & \frac{3^{N+1}D^{j+2N+2}(j+N)!}{N!N!(N+1)!}(\xi\eta)^N\gamma_{j+1,N}(j+4N+4) = \frac{3^{N+2}\gamma_{j,N+1}D^{j+2N+2}(j+N)!}{N!N!(N+1)!}(\xi\eta)^N
\end{aligned}
$$

Box I.

For each pair (I, J) with unknown $\bar{k}_{I,J}$ (i.e., the points that are not on the boundaries with known boundary conditions), (B.1) relates the kernel function and its time derivative of computational points in the form of an ODE. The new indexing $r = (2N - j)(j - 1) + i$ will vectorize the array $\bar{k}_{i,j}$ into κ_r and the set of $N^2 - 3N + 2$ ODE's can be written as in (29) with $H_r(t)$ arising from the values of the kernel function on boundaries with known boundary conditions as well as the term $\frac{4\alpha}{l^2}\left[f(\xi_I, t) - f(\eta_J, t)\right]$. Note that matrices A and $B(t)$ in (29) are in the lower triangular form.

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